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LETTER TO THE EDITOR

On the relation between the magnetic constant and the photon distribution function in a medium

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Abstract. The linear response theory is used to obtain the exact relation between the magnetic constant of a medium and the equilibrium photon distribution function. The magnetic permeability of the degenerate ideal Bose-gas of charged particles is examined.

The study of magnetic permeability is one of the most important tasks in the linear electrodynamics of charged-particle systems. In this case, in connection with the development of the theory for the strong-interaction systems, particular importance is attached to the general relations for correlation functions which, having been satisfied, would appear to corroborate the self-consistency of the theory. In the present work, the exact relation between the magnetic constant of a medium and the equilibrium photon distribution function is obtained in terms of the linear response theory.

The linear magnetic permeability $\mu(\boldsymbol{k}, \omega)$ of a homogeneous and isotropic medium is related to the transversal, $\varepsilon^{\mathrm{tr}}(\boldsymbol{k}, \omega)$, and longitudinal, $\varepsilon^{\mathrm{l}}(\boldsymbol{k}, \omega)$, dielectric permittivity (Silin and Rukhadze 1961) as

$$1 - \mu^{-1}(\boldsymbol{k}, \omega) = (\omega^2/c^2k^2)(\varepsilon^{\mathrm{tr}}(\boldsymbol{k}, \omega) - \varepsilon^{\mathrm{l}}(\boldsymbol{k}, \omega)) \tag{1}$$

where c is the speed of light. At finite values of the wave vector, $\boldsymbol{k}$, the function $\varepsilon^{\mathrm{l}}(\boldsymbol{k}, \omega)$ does not exhibit any singularities in the limit $\omega \to 0$, so we find for the static magnetic permeability $\mu(\boldsymbol{k}, 0)$:

$$1 - \mu^{-1}(\boldsymbol{k}, 0) = \lim_{\omega\to 0}(\omega^2/c^2k^2)\varepsilon^{\mathrm{tr}}(\boldsymbol{k}, \omega). \tag{2}$$

Therefore, the experimentally measurable magnetic constant $\bar{\mu}$ of a medium is

$$\bar{\mu} = \lim_{k\to 0}\mu(\boldsymbol{k}, 0) = (1 - \lim_{k\to 0}\lim_{\omega\to 0}(\omega^2/c^2k^2)\varepsilon^{\mathrm{tr}}(\boldsymbol{k}, \omega))^{-1}. \tag{3}$$

In terms of the linear response theory, the function $\varepsilon^{\mathrm{tr}}(\boldsymbol{k}, \omega)$ takes the form (Bobrov and Trigger 1988)

$$\varepsilon^{\mathrm{tr}}(\boldsymbol{k}, \omega) = (c^2k^2/\omega^2)[1 + (4\pi/k^2)(D^{\mathrm{R}}(\boldsymbol{k}, \omega))^{-1}] \tag{4}$$

where

$$D^{\mathrm{R}}_{\alpha\beta}(\boldsymbol{k}, \omega) = (\delta_{\alpha\beta} - k_\alpha k_\beta/k^2)D^{\mathrm{R}}(\boldsymbol{k}, \omega). \tag{5}$$

$D^{\mathrm{R}}_{\alpha\beta}(\boldsymbol{k}, \omega)$ is the Fourier component of the retarded photon Green function

$$D^{\mathrm{R}}_{\alpha\beta}(\boldsymbol{r}_1 - \boldsymbol{r}_2, t) = -(\mathrm{i}/\hbar)\theta(t)\,\mathrm{Sp}\,\hat{F}[\hat{A}_\alpha(\boldsymbol{r}_1, t), \hat{A}_\beta(\boldsymbol{r}_2, 0)] \tag{6}$$

$\hat{F}$ is the statistical Gibbs operator of a quantum–electrodynamic system which is a

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combination of photons and non-relativistic particles; $\hat{A}_\alpha(\boldsymbol{r}, t)$ is the operator of vector potential in the Heisenberg representation corresponding to a quantised electromagnetic field (Akhiezer and Peletminsky 1977):

$$\hat{A}_\alpha(\boldsymbol{r}, t) = c \sum_{k\lambda} \left(\frac{2\pi\hbar}{ckV}\right)^{1/2} [\boldsymbol{e}^{(\lambda)}_{k\alpha} \hat{a}_{k\lambda} \exp(\mathrm{i}kr) + \boldsymbol{e}^{(\lambda)*}_{k\alpha} \hat{a}^+_{k\lambda} \exp(-\mathrm{i}kr)] \tag{7}$$

where $\hat{a}^+_{k\lambda}$ ($\hat{a}_{k\lambda}$) is the creation (annihilation) operator for a photon with momentum $\hbar \boldsymbol{k}$ and polarisation $\lambda = 1.2$; $\boldsymbol{e}^{(\lambda)}_{k\alpha}$ is the photon polarisation vector satisfying the conditions

$$\sum_\alpha \boldsymbol{e}^{(\lambda)}_{k\alpha} k_\alpha = 0 \qquad \sum_\lambda \boldsymbol{e}^{(\lambda)}_{k\alpha} \boldsymbol{e}^{(\lambda)*}_{k\beta} = \delta_{\alpha\beta} - k_\alpha k_\beta / k^2 \tag{8}$$

where V is the system volume; $\theta(t) = 1$ at $t > 0$ and 0 at $t < 0$.

Substituting (4) in (2), we obtain

$$\mu(\boldsymbol{k}, 0) = -(k^2/4\pi) D^{\mathrm{R}}(\boldsymbol{k}, 0). \tag{9}$$

Using the spectral representation of the function $D^{\mathrm{R}}(\boldsymbol{k}, \omega)$, we can readily verify that

$$D^{\mathrm{R}}(\boldsymbol{k}, 0) = D^{\mathrm{T}}(\boldsymbol{k}, 0) < 0 \tag{10}$$

where $D^{\mathrm{T}}(\boldsymbol{k}, \mathrm{i}\Omega_n)$ is the photon temperature Green function corresponding to the retarded Green function $D^{\mathrm{R}}(\boldsymbol{k}, \omega)$ (Abrikosov *et al* 1962), $\Omega_n = 2\pi n T$, $n = 0, 1, 2, \ldots$, and T is the temperature of the medium.

The well known proposition concerning the positivity of static magnetic permeability (Kirzhnitz 1987) follows immediately from relations (9) and (10).

If the magnetic constant $\bar{\mu}$ is a non-zero finite value (as is the case for normal systems), the function $D^{\mathrm{R}}(\boldsymbol{k}, 0)$ will exhibit the singularity $1/k^2$ at $k \to 0$. In this connection we shall examine the $D^{\mathrm{R}}(\boldsymbol{k}, 0)$ behaviour for the case of small wave vectors, $\boldsymbol{k}$.

The results obtained by Perel' and Eliashberg (1962) can readily be used to verify that

$$T \sum_{\Omega_n} D^{\mathrm{T}}(\boldsymbol{k}, \mathrm{i}\Omega_n) = \hbar \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \coth\left(\frac{\hbar\omega}{2T}\right) \operatorname{Im} D^{\mathrm{R}}(\boldsymbol{k}, \omega). \tag{11}$$

On the other hand, from the spectral representation of the Green function $D^{\mathrm{R}}(\boldsymbol{k}, \omega)$ allowing for the relations (6)–(8), it follows immediately that

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \coth\left(\frac{\hbar\omega}{2T}\right) \operatorname{Im} D^{\mathrm{R}}(\boldsymbol{k}, \omega) = -(4\pi c/k)(n(\boldsymbol{k}) + \tfrac{1}{2}) \tag{12}$$

where $n(\boldsymbol{k})$ is the equilibrium photon distribution function in the medium

$$n(\boldsymbol{k}) = \mathrm{Sp}\, \hat{F} \hat{a}^+_{k\lambda} \hat{a}_{k\lambda} \tag{13}$$

whence

$$T \sum_{\Omega_n} D^{\mathrm{T}}(\boldsymbol{k}, \mathrm{i}\Omega_n) = -(4\pi\hbar c/k)(n(\boldsymbol{k}) + \tfrac{1}{2}). \tag{14}$$

The transversal permittivity of the medium, $\varepsilon^{\mathrm{tr}}(\boldsymbol{k}, \omega)$, is a finite value in the long-wave limit $k/\omega \to 0$ (Silin and Rukhadze 1961). Therefore, the limit $\lim_{k\to 0} D^{\mathrm{T}}(\boldsymbol{k}, \mathrm{i}\Omega_n)$ is a finite value at $\Omega_n \neq 0$ (see equation (4)). So,

$$\lim_{k\to 0} \sum_{\Omega_n} D^{\mathrm{T}}(\boldsymbol{k}, \mathrm{i}\Omega_n) = \lim_{k \to 0} D^{\mathrm{T}}(\boldsymbol{k}, 0). \tag{15}$$

As a result, using the relations (3), (10), and (15), we obtain the final expression for the magnetic constant, $\bar{\mu}$, of a medium through the equilibrium photon distribution function

$$\bar{\mu} = \lim_{k \to 0}(\hbar ck/T)(h(k) + \tfrac{1}{2}). \tag{16}$$

In the special case of the free radiation field, we have

$$n(k) = (\exp(\hbar ck/T) - 1)^{-1} \tag{17}$$

hence $\bar{\mu} = 1$.

As noted above, the result (16) is valid in the case of normal systems with $\bar{\mu} \neq 0$.

A medium with a zero-value magnetic constant $\bar{\mu}$ may be exemplified by a system composed of photons and of the ideal gas of charged Bose-particles of density n, spin $S = 0$, charge ze, and mass m at $T < T_0$, where $T_0 = 3.31(\hbar^2 n^{2/3}/m)$ is the Bose-condensation temperature.

In the case of weak photon–particle interaction, the function $\varepsilon^{\rm tr}(\boldsymbol{k}, \omega)$ is of the form (Bobrov and Trigger 1988)

$$\varepsilon^{\rm tr}(\boldsymbol{k}, \omega) = 1 - \omega_{\rm p}^2/\omega^2 - (2\pi/\omega^2)(\delta_{\alpha\beta} - k_\alpha k_\beta/k^2)G^{\rm R}_{\alpha\beta}(\boldsymbol{k}, \omega) \tag{18}$$

where $\omega_{\rm p} = (4\pi z^2 e^2 n/m)^{1/2}$ is the plasma frequency, $G^{\rm R}_{\alpha\beta}(\boldsymbol{k}, \omega)$ is the Fourier component of the retarded tensor Green function of currents

$$G^{\rm R}_{\alpha\beta}(\boldsymbol{r}_1 - \boldsymbol{r}_2, t) = -({\rm i}/\hbar)\theta(t)\, SP\, \hat{F}_0[\hat{j}_\alpha(\boldsymbol{r}_1, t), \hat{j}_\beta(\boldsymbol{r}_2, 0)] \tag{19}$$

where $\hat{F}_0$ is the statistical Gibbs operator of the charged-particle system and $\hat{j}_\alpha(\boldsymbol{r}, t)$ is the operator of electric current density in the Heisenberg representation when the electromagnetic field is absent.

In the case of the ideal gas of charged particles with spin $S = 0$, the tensor Green function of currents is (Akhiezer and Peletminsky 1977)

$$(\delta_{\alpha\beta} - k_\alpha k_\beta/k^2)G^{\rm R}_{\alpha\beta}(\boldsymbol{k}, \omega) = \frac{z^2 e^2 \hbar^2}{m^2 V}\sum_p \frac{f_{p-k/2} - f_{p+k/2}}{\hbar\omega + \varepsilon_{p-k/2} - \varepsilon_{p+k/2} + {\rm i}\delta}(p^2 - (k_\alpha p_\alpha)^2/k^2) \tag{20}$$

where $f_p = S_p \hat{F}_0 \hat{b}_p^+ \hat{b}_p$; $\hat{b}_p^+$ ($\hat{b}_p$) is the creation (annihilation) operator for a particle with momentum $\hbar p$, $\varepsilon_p = \hbar^2 p^2/2m$, $\delta = +0$.

At $T < T_0$ we have

$$f_p = N_0 \delta_{p,0} + f_p^{\rm T}(1 - \delta_{p,0}) \tag{21}$$

where $N_0 = N(1 - (T/T_0)^{3/2})$ is the number of particles with energy, $\varepsilon_p = 0$ and N is the total mean number of particles

$$f_p^{\rm T} = (\exp(\varepsilon_p/T) - 1)^{-1} \tag{22}$$

with

$$\int \frac{{\rm d}^3 p}{(2\pi)^3} f_p^{\rm T} = n\left(\frac{T}{T_0}\right)^{3/2}. \tag{23}$$

Therefore,

$$(\delta_{\alpha\beta} - k_\alpha k_\beta/k^2)G^{\rm R}_{\alpha\beta}(\boldsymbol{k}, \omega) = \frac{z^2 e^2 \hbar^2}{m^2}\int \frac{{\rm d}^3 p}{(2\pi)^3} \frac{f^{\rm T}_{p-k/2} - f^{\rm T}_{p+k/2}}{\hbar\omega + \varepsilon_{p-k/2} - \varepsilon_{p+k/2} + {\rm i}\delta}(p^2 - (p_\alpha k_\alpha)^2/k^2). \tag{24}$$

It can be readily verified that

$$(2\pi/\omega^2)(\delta_{\alpha\beta} - k_\alpha k_\beta/k^2)G^{\rm R}_{\alpha\beta}(\boldsymbol{k}, \omega) \ll 1 \tag{25}$$

at $T \ll T_0$, so the transversal permittivity of the ideal Bose-gas of zero-spin particles at $T \ll T_0$ is

$$\varepsilon^{\mathrm{tr}}(\boldsymbol{k}, \omega) \simeq 1 - \omega_{\mathrm{p}}^2/\omega^2 \tag{26}$$

whence

$$\mu(\boldsymbol{k}, 0) = c^2k^2/(c^2k^2 + \omega_{\mathrm{p}}^2) \tag{27}$$

$$D^{\mathrm{R}}(\boldsymbol{k}, 0) = -4\pi c^2/(c^2k^2 + \omega_{\mathrm{p}}^2). \tag{28}$$

The relations (27) and (28) correspond to the case of an ideal London's superconductor. In this case we have

$$\begin{aligned} n(\boldsymbol{k}) + \tfrac{1}{2} &= -\frac{k}{4\pi c}\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\coth\left(\frac{\hbar\omega}{2T}\right) \operatorname{Im} D^{\mathrm{R}}(\boldsymbol{k}, \omega) \\ &= -\frac{k}{2\pi c}\int_{0}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\coth\left(\frac{\hbar\omega}{2T}\right) \operatorname{Im} D^{\mathrm{R}}(\boldsymbol{k}, \omega) \\ &\simeq kc\int_{0}^{\infty} \frac{\mathrm{d}\omega}{\omega^2}\coth\left(\frac{\hbar\omega}{2T}\right)\delta\left(1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2} - \frac{c^2k^2}{\omega^2}\right) \\ &= \frac{kc}{2\omega(\boldsymbol{k})}\coth\left(\frac{\hbar\omega(\boldsymbol{k})}{2T}\right) \end{aligned} \tag{29}$$

where $\omega(\boldsymbol{k}) = (c^2k^2 + \omega_{\mathrm{p}}^2)^{1/2}$.

Thus, the relation (15) is not satisfied rigorously in the case of the degenerate ideal Bose-gas. The relation (16), however, appears to be valid, as before.

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